FINAL REPORT OF THE WORK DONE ON THE PROJECT

"On L^1-Convergence of Trigonometric series with Special coefficients"

F.No. 42-3/2013(SR)

Submitted to

UNIVERSITY GRANTS COMMISSION
BHADURSHAH ZAFAR MARG
NEW DELHI -110002

Submitted by

Dr. Jatinderdeep Kaur
(Principal Investigator)

THAPAR INSTITUTE
OF ENGINEERING & TECHNOLOGY
(Deemed to be University)

Introduction

It is well known that if a trigonometric series converges in L^1-metric to a function $f \in L^1(T)$ then it is the Fourier series of the function *f*. In 1932, Risez gave a counter example to show that in L^1-metric, the converse of the above said result does not hold good. This motivated various authors to study the L^1-convergence of trigonometric series with special coefficients. Integrability and L^1-convergence of trigonometric series with special coefficients have been studied by number of authors. The work on this topic was introduced by Young W.H.(1913) and Kolmogorov A.N. by taking classes of convex sequences($\Delta^2 a_n \geq 0$) and quasi convex sequences $\left(\sum_{n=1}^{\infty} n \left| \Delta^2 a_n \right| < \infty \right)$ respectively. They proved the following theorem:

Theorem A: If $\{a_k\}$ is a convex(or quasi convex) null sequence, then

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx \in L^1[0,\pi].$$

Teljakovskii S.A. (1973) considered another class S (introduced by Sidon (1939)) of L^1-convergence and proved the following result:

Theorem B: Let the cosine series $\sum_{k=1}^{\infty} a_k \cos kx$ belong to class S. Then it is a Fourier series and the following relation holds:

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx \in L^1[0,\pi].$$

During investigations, some authors introduced modified trigonometric sums, as these sums approximate their limits better than the classical trigonometric series in the sense that these sums converge in L^1-metric to the sum of trigonometric series whereas the classical series itself may not. Garret J.W. and Stanojevic (1976), Kumari S. and Ram B. (1987), Chen C.P. (1987), Kaur K., Bhatia S.S. and Ram B. (2004), Kaur J., Bhatia S.S. and Ram B. (2008) and many others have introduced new modified trigonometric sums and studied their L^1-

convergence under various classes of coefficient sequences.

L^1-convergence of complex trigonometric series has been studied by various authors such as Stanojevic C.V. and Stanojevic V.B.(1987), Bhatia S.S. and Ram B. [(1993), (1995)], Tomovskii (2003), Kaur J. and Bhatia S.S. (2010) and many others for various classes of complex sequences.

Integrability and L^1-convergence of double trigonometric series has been studied by various authors. In 1991, Moricz F. extended the classical theorems of Young W.H. (1913) and Kolmogorov A.N. (1923) from one dimension cosine and sine series to two dimensional cosine and sine series by considering the special cases. After that many authors have studied the L^1-convergence of double trigonometric series.

Objectives: The objectives are given below:

1. To obtain the complex form of modified trigonometric sums introduced by X. Z. Krasniqi [2009] and to study the necessary and sufficient condition for the integrability and L^1-convergence of complex trigonometric series.

2. To extend the results obtained in objective 1 as well as that of Krasniqi [2009] from one dimension to two dimensions.

3. To generalize the results obtained in above two objectives for multidimensional trigonometric series (dim>2).

4. To obtain new modified trigonometric sums for several variables and to study the L^1-convergence of multiple trigonometric series using the newly defined modified sums for several variables.

5. We will try to apply the results obtained in above objectives to those problems where it is necessary to have estimate for the integral of a function in terms of Fourier coefficients.

Methodology:

1. Literature survey.
2. To introduce new modified trigonometric sums in real and complex form.
3. To obtain new classes of coefficient sequences to study the integrability and L^1-convergence of trigonometric series in one and two dimensions.

Work Done:

As per objectives and work plan, Firstly, we have done literature survey. During literature survey , we found that modified trigonometric sums and different conditions on coefficients are tool to study the L^1 -convergence of trigonometric series in one as well as in two dimensions.

we have completed first objective i.e. to obtain the complex form of modified trigonometric sums introduced by X. Z. Krasniqi [2009] and to study the necessary and sufficient condition for the integrability and L^1-convergence of complex trigonometric series.

The complex form of modified cosine and sine trigonometric sums

$$H_n(x) = -\frac{1}{2\sin x}\sum_{k=0}^{n}\sum_{j=k}^{n}\Delta[(a_{j-1}-a_{j+1})\cos jx]$$

and

$$G_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}\Delta[(a_{j-1}-a_{j+1})\sin\ jx]$$

is

$$g_n(C,t) = S_n(C,t) + \frac{i}{2\sin t}\left[\begin{array}{r} c_n e^{i(n+1)t} - c_{-n}e^{-i(n+1)t} + c_{n+1}e^{\text{int}} - c_{-(n+1)}e^{-\text{int}} \\ +(n+1)(c_n - c_{n+2})e^{i(n+1)t} + (n+1)(c_{-(n+2)} - c_{-n})e^{-i(n+1)t} \end{array}\right]$$

and have proved the convergence of complex trigonometric series under class J* of complex coefficients in metric space L .

Main Result. *Let* $\hat{c}_n \in J^*$. *Then there exists* $\hat{f}(t)$ *such that*

(i) $\lim_{n\to\infty} \hat{g}_n(\hat{C},t) = \hat{f}(t)$ *for* $|t| \in (0,\pi]$,

(ii) $\hat{f}(t) \in L^1(0,\pi]$ *and* $\left\|\hat{g}_n(\hat{C},t) - \hat{f}(t)\right\|_{L^1} = o(1)$ *as* $n \to \infty$,

(iii) $\left\|\hat{S}_n(\hat{f},t) - \hat{f}(t)\right\|_{L^1} = o(1)$ *as* $|n| \to \infty$.

This result is published in **IJETCAS, Vol. 11, Issue 2, 110-114 (2015).**

Along with this, due to advancement in Fuzzy theory, we have also extended the classical results on Fourier Analysis to fuzzy Fourier analysis. We have obtained necessary and sufficient conditions for fuzzy integrability and L^1-convergence of fuzzy trigonometric series under more generalized conditions on fuzzy coefficients.

New classes of fuzzy coefficients are introduced as follows:

- A sequence $\{a_k\} \in w(F)$ is said to be decreasing sequence if $a_{k+1} \prec a_k$ *i.e.* $(a_k)_\lambda^- < (a_{k+1})_\lambda^-$ and $(a_k)_\lambda^+ > (a_{k+1})_\lambda^+$.
- A sequence $\{a_k\} \in w(F)$ is said to be fuzzy null sequence if $\lim_{k\to\infty} a_k = [0]_\lambda$ with respect to the level sets. *i.e.* $\left\{\left(\lim_{k\to\infty}(a_k)_\lambda^- = 0_\lambda^-, \lim_{k\to\infty}(a_k)_\lambda^+\right) \mid 0 \le \lambda \le 1\right\}$
- A decreasing sequence $\{a_k\} \in w(F)$ is said to belong to class **BV(F)** with respect to the level if $\{a_k\}$ is a fuzzy null sequence and the series $\sum_{\oplus\, k=0}^{\infty} |\Delta a_k|$ is convergent.
- A decreasing sequence $\{a_k\} \in w(F)$ is said to belong to class $K_P(F)$ (for $\lambda \in [0,1]$) if $\{a_k\}$ is a fuzzy null sequence and the series

$$\sum_{\oplus\, m=1}^{\infty} 2^{m/q}\left(\sum_{\oplus\, k\in I_m} |\Delta a_k|^p\right)^{1/p} \quad , \quad \text{for some } p > 1$$

Let

$$\frac{1}{2}a_0 \oplus \sum_{\oplus\, k=1}^{\infty} a_k \cos kx$$

and $\sum_{\oplus\, k=1}^{\infty} b_k \sin kx$ be fuzzy trigonometric cosine and sine series.

Theorem 2: Let $\{b_k\}$ be a sequence of fuzzy coefficients belonging to class **BV(F)**, then

(i). the series converges $\sum_{\oplus\, k=1}^{\infty} b_k \sin kx$ in L^1 - norm if and only if $\sum_{\oplus\, k=1}^{\infty} \frac{|b_k|}{k} < \infty$.

(ii). if $g^t \in L^1(0,\pi)$, then $\sum_{\oplus\, k=1}^{\infty} b_k \sin kx$ is the Fourier series of g^t.

This result has been published in the **Proceedings of International Conference on Mathematical Sciences, held at Sathyabama University, Chennai, July 16-18, 2014 (Elsevier).**

Theorem 3: If $\{a_k\}$ is a sequence of fuzzy coefficients belonging to class $K_p(F)$**,** then

(i). $f^t \in L^1(0,\pi)$**.**

(ii). series $\frac{1}{2}a_0 \oplus \sum_{\oplus\, k=1}^{\infty} a_k \cos kx$ is the Fourier series of f^t.

(iii). the series $\frac{1}{2}a_0 \oplus \sum_{\oplus\, k=1}^{\infty} a_k \cos kx$ converges to fuzzy valued function f^t in $L^1(0,\pi)$-norm if and only if $\left\{ \lim_{n\to\infty} a_n \ln n = [0]_\lambda \;\middle|\; 0 \le \lambda \le 1 \right\}$

Theorem 4: If $\{b_k\}$ is a sequence of fuzzy coefficients belonging to class $K_p(F)$, then

(i). $g^t \in L^1(0,\pi)$ if and only if $\sum_{\oplus\, k=1}^{\infty} \frac{|b_k|}{k} < \infty$.

(ii). if $g^t \in L^1(0,\pi)$, then $\sum_{\oplus\, k=1}^{\infty} b_k \sin kx$ is the Fourier series of g^t.

(iii). the series $\sum_{\oplus\, k=1}^{\infty} b_k \sin kx$ converges to the fuzzy valued function g^t in $L^1(0,\pi)$-norm if and only if $\left\{ \lim_{n\to\infty} b_n \ln n = [0]_\lambda \;\middle|\; 0 \le \lambda \le 1 \right\}$.

These results has been published in **Journal of Nonlinear Science and Applications 8(2015) 23-39 (SCI).**

After that we have introduced modified trigonometric cosine and sine sums as

$$f_n(t) = \sum_{k=1}^{n} \left(\frac{a_{k+1}}{k+1} + \sum_{j=k}^{n} \Delta^2 \left(\frac{a_j}{j} \right) \right) k \cos kt \qquad (a_0 = a_1 = a_2 = 0)$$

and

$$g_n(t) = \sum_{k=1}^{n} \left(\frac{b_{k+1}}{k+1} + \sum_{j=k}^{n} \Delta^2 \left(\frac{b_j}{j} \right) \right) k \sin kt \qquad (b_1 = b_2 = 0)$$

and a criterion for the summability and L^1-convergence of these modified sums is obtained. Also an application is given to illustrate the main result.

Theorem. *If $\{a_k\}$ is a generalized semi-convex, then $f_n(t)$ converges to $f(t)$ in L^1-metric if and only if* $\lim_{n\to\infty} a_n \log n = 0$.

Theorem. *If $\{b_k\}$ is a generalized semi-convex, then $g_n(t)$ converges to $g(t)$ in L^1-metric if and only if* $\lim_{n\to\infty} b_n \log n = 0$.

This paper “Convergence and Summability of Fourier Sine and Cosine series with its applications” has been published in **Proceedings of National Academy of Sciences, India, Sect A Phys. Sci.** **DOI 10.1007/s40010-017-0471-5**

In 2016, we introduced a new modified double trigonometric sine sum which is an extension of Xhevat modified trigonometric sine sum in one dimension and study its integrability and L1 convergence of double trigonometric cosine series under new class.

Theorem 1. If a double sequence $\{a_{jk}\}$ belongs to the class S_{jk}. Then

(i) $\lim_{m,n\to\infty} g_{mn}(x,y) = f(x,y)$ exists.

(ii) $f \in L^1(Q)$.

(iii) $\| g_{mn} - f \| \to 0 \;\; as \;\; m+n \to \infty.$

List of Publications

Publications In Journals

1. Sandeep Kaur and Jatinderdeep Kaur, *"Integrability and L^1-convergence of fuzzy trigonometric series with special coefficients"*, J. Nonlinear Sci. Appl. 8 (2015), 23–39. (SCI)

2. Jatinderdeep Kaur, Sandeep Kaur and S.S. Bhatia, *"Lebesgue Convergence of Modified Complex Trigonometric Sums"*, International Journal of Emerging Technologies in Computational and Applied Sciences, 11(2) 2015, 110-114.

3. Sandeep Kaur and Jatinderdeep Kaur, *"On L^1- convergence of modified cosine sums"* Mathematical Sciences International Research Journal, 4(1)(2015), 104-107.

4. Sandeep Kaur, Jatinderdeep Kaur and S.S. Bhatia *"Lebesgue convergence of modified cosine sum of fuzzy valued functions"* Mathematical Sciences International Research Journal, 4(2)(2015), 65-69.

5. Jatinderdeep Kaur, *"Coefficients of Some Double trigonometric Cosine and Sine Series"*, International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering, 9(11)(2015), 550-553.

6. Sandeep Kaur, Jatinderdeep Kaur and S.S. Bhatia, *"Convergence and Summability of Fourier sine and cosine series with its Applications"*, Proceedings of National Academy of Sciences, India, Sect A Phys. Sci. DOI 10.1007/s40010-017-0471-5.

7. Sandeep Kaur, Jatinderdeep Kaur and S.S. Bhatia, *"Integrability and L^1-convergence of double trigonometric cosine series"*. (Communicated)

Presented and Published in Proceedings of Conference:

1. Sandeep Kaur and Jatinderdeep Kaur, *" Lebesgue integrability of fuzzy trigonometric sine series"* published in Proceedings of International Conference on Mathematical Sciences at Sathyabama University Chennai (July 17-19, 2014).

2. Sandeep Kaur and Jatinderdeep Kaur, "*On L^1- convergence of modified cosine sums"* International Conference on Advances in Mathematical Sciences-2015 (ICAM-2015) at Khalsa College Patiala (March 19-21, 2015).

3. Sandeep Kaur, Jatinderdeep Kaur and S.S. Bhatia "*Lebesgue convergence of modified cosine sum of fuzzy valued functions"* International Conference on Mathematics-2015(ICM-2015) at University of Kerela, Thiruvanthapuram, Kerela, India (November 26-28, 2015).